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LETTER TO THE EDITOR

Correspondence between neural threshold networks and Kauffman Boolean cellular automata

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Abstract. For an asymmetric version of the McCulloch-Pitts neural network and for Kauffman's infinite-range Boolean network model, the time evolution of the Hamming distances between two different initial configurations are compared in the thermodynamic limit. It is shown that in both models phase transitions occur for corresponding values of the transition parameters and that their Hamming distances can have the same time evolution leading to quantitatively the same dynamics, as known from time-dependent Landau theory for phase transitions.

It has often been assumed that neural network models having their roots in the pioneering work of McCulloch and Pitts [1] share many common features with Kauffman's random Boolean network [2], a model of biological evolution, which has recently found renewed interest in the field of disordered cellular automata [3]. There are numerical arguments pointing to the existence of close formal relationships with this model [4, 5]. From an analytical point of view it is plausible that a finite sequence of Boolean operations can be defined in terms of a neural network with weighted interactions and suitable thresholds. This property is well known from classical theories of networks, where the exclusive OR function can be realised by a functional threshold network.

The microscopic dynamics of both models consisting of N randomly interconnected cells, formal neurons or Ising spins, which can take only two possible values ($\sigma_i = +1$ or $\sigma_i = -1$), may be described in the following form:

$$\sigma_i(t+\tau) = f_i(\sigma_{n_1(i)}(t), \ldots, \sigma_{n_K(i)}(t)) \qquad i = 1, \ldots, N \tag{1}$$

with $n_j(i) \in \{1, \ldots, N\}$ and $j = 1, \ldots, K$. Each cell is supposed to receive exactly K randomly chosen distinct inputs and the quantity τ may be interpreted as the delay time for signal transmission from one cell to another. In the Kauffman model the functions f_i are determined by randomly chosen Boolean functions, whereas in models of threshold automata and neural networks, the functions f_i are defined by the rule

$$f_i = \operatorname{sgn}\left(\sum_j c_{ij}\sigma_j(t) + h\right) \qquad i = 1, \ldots, N \tag{2}$$

where j runs over the K inputs of cell i. Here, the weights of the synaptic efficiencies among the cells are represented by the quantities c_{ij} sampled randomly from a given

distribution $\rho(c_{ij})$, the parameter h defining a threshold value supposed to be the same for all cells of the network. Note that the coupling coefficients are usually not symmetric. According to (1) the deterministic dynamics of these networks evolve synchronously in discrete time and the state of the network at time t is given by the state vector $\boldsymbol{\sigma}(t)$.

Since phase space consists only of a finite number of 2^N distinct states, the system will inevitably evolve to attractors consisting of limit cycles or fixed points after having passed through a transient phase. For both models it has been convincingly demonstrated by analytical results as well as by computer simulations that two phases may exist, a chaotic phase and a frozen phase, depending on the network parameters [3-6]. In the ordered phase the average period increases with a power of the total number of cells, whereas in the chaotic phase the mean cycle length increases exponentially with N [3-5]. At the critical point the behaviour is intermediate [4]. The crucial quantity for the theoretical prediction of the existence of these two phases is the time evolution of the normalised Hamming distance $H_K(t)$, representing the fraction of spins being different in two states $\boldsymbol{\sigma}^{(1)}(t)$ and $\boldsymbol{\sigma}^{(2)}(t)$. For a symmetric distribution of the coupling coefficients an analytic expression for $H_K(t)$ can be derived for the neural network model in the thermodynamic limit under the assumption that K remains finite. Following closely Derrida's arguments [7] this quantity takes the form of a polynomial spline function of order K [4]

$$H_K(t+\tau) = F_K(H_K(t)) = \sum_{\nu=1}^{K} (-1)^{\nu+1} \binom{K}{\nu} a_\nu H_K^\nu(t) \tag{3}$$

with

$$a_\nu = 1 + \sum_{m=1}^{\nu} (-1)^m \binom{m}{\nu} I_m^{(K)} \qquad \nu = 1, \ldots, K \tag{4}$$

and

$$I_m^{(K)}(\rho, h) = \int \ldots \int \mathrm{d}x_1 \ldots \mathrm{d}x_K \, \rho(x_1) \ldots \rho(x_k) \times \theta\{(x_{m+1} + \ldots + x_K + h)^2 - (x_1 + \ldots + x_m)^2\} \tag{5}$$

where $\theta(x)$ represents the Heaviside step function. The existence of a phase transition then depends uniquely on the nature of the fixed point $H_K \equiv 0$. If it is attractive, two initial configurations differing by an infinitesimal fraction of spins will become almost identical, whereas if it is repulsive, they will produce diverging trajectories. A possible critical point or critical line is then completely determined for $a_1 \neq 0$ by the equation

$$\left.\frac{\mathrm{d}F_K}{\mathrm{d}H_K}\right|_{H_K \equiv 0} = Ka_1 = 1. \tag{6}$$

Hence, any critical behaviour, depending only on the first-order coefficient a_1, is governed by the relation $a_1 = 1/K$, representing a necessary condition for the existence of a critical point. It should be stressed that the above derivation also holds for the Kauffman model. In this case the strictly non-negative coefficients a_ν in (4) are specialised to be constant as shown by Derrida and Pomeau [6] taking the value $2p(p-1)$, p being the probability that the randomly chosen functions f_i are unity. This is in contrast to the neural network model, where the coefficients usually differ by definition of the K-dimensional integrals in (5). For $H_K(t) \ll 1$, equation (3) can be written as

$$H_K(t+1) = H_K(t) + K(a_1 - 1/K)H_K(t) - a_2 \binom{K}{2} H_K^2(t) + \mathrm{O}(H_K^3(t)) \tag{7}$$

the structure being reminiscent of time-dependent Landau theory for phase transitions. Under the assumption that t is not too large the behaviour for $H_K(t)$ in continuous time is well described by the solution of the corresponding linearised differential equation for $a_1 \neq 1/K$:

$$\dot{H}_K(t) = K(a_1 - 1/K)H_K(t) \tag{8a}$$

$$H_K(t) = H_K(t=0)\exp[K(a_1 - 1/K)t]. \tag{8b}$$

Thus, for $a_1 > 1/K$, where the fixed point $H_K \equiv 0$ is unstable, $H_K(t)$ increases exponentially for small t finally reaching its asymptotic value given by a stable fixed point $H_K \neq 0$ of (3), whereas for $a_1 < 1/K$, $H_K(t)$ decreases exponentially to zero. At the critical point $a_1 = 1/K$ the second term in (7) vanishes and the continuous-time behaviour of $H_K(t)$ can be determined from the solution of the first-order differential equation

$$\dot{H}_K(t) = -a_2\binom{K}{2}H_K^2(t) \tag{9a}$$

$$H_K(t) = a_2\binom{K}{2}\frac{H_K(t=0)}{H_K(t=0)t+1}. \tag{9b}$$

Thus, at the critical point the asymptotic time behaviour follows a simple inverse-t law.

As has been demonstrated earlier for $K \leq 2$, the neural network, as well as the Kauffman model, strictly remains in the frozen phase since the fixed point $H_K \equiv 0$ is always stable due to the fact that the coefficient a_1 never exceeds the value $1/K$ [3-6].

For the neural network model it is interesting to consider the special case of zero threshold where additional symmetries are introduced into the model and where we expect the highest degree of disorder provided all the other network parameters are fixed. Apart from the trivial fixed points $H_K = 0$ and $H_K = 1$ the symmetry condition, which can be easily derived from (5),

$$I_{K-\nu}^{(K)} = 1 = I_\nu^{(K)} \qquad \nu = 1, \ldots, K-1 \tag{10}$$

guarantees that $H_K = 0.5$ is always a fixed point of (3). In the frozen phase the fixed points $H_K = 0$ and $H_K = 1$ are stable, whereas $H_K = 0.5$ is unstable. In the chaotic phase these fixed points change their stability from attraction to repulsion and $H_K = 0.5$ becomes a stable fixed point implying that in the chaotic phase the memory of initial conditions is *completely* eliminated after sufficiently long time t. Making use of the symmetry condition (10) one can show that at the critical point for $K = 3$ and $K = 4$ the higher-order coefficients $a_{\nu>1}$ vanish. Hence, $H_K(t)$ remains a constant equal to the initial value $H_K(t=0)$ at this higher-order critical point. However, for $K \geq 5$ the higher-order coefficients are in general non-zero. Note that for the parameter choice

$$2p(p-1) = 1 - I_1^{(K)} \tag{11}$$

the time evolution for the Hamming distance for $H_K(t) \ll 1$ (7) has the same coefficient a_1 in the Kauffman model and the neural network model, giving rise to the same dynamics for small initial distances. Moreover, for $K = 1$ the time evolution for any arbitrary Hamming distance $H_1(t)$ is evidently the same.

In order to give such agreement for larger values of the connectivity parameter K, the Kauffman model has to be suitably generalised such that higher-order coefficients a_ν are all different and agree with those of the neural network model. Since there exist 2^{2^K} distinct Boolean functions one might give differing weights to different functions.

Another possibility is to dilute the connectivity of the system such that a fraction x_i of the cells receive exactly i inputs for $i = 1, \ldots, K$. Generalising the result of Derrida and Pomeau [6] for the time evolution of the Hamming distance $H_K(t)$ to arbitrary control parameters p leads to the following equation:

$$H_K(t+\tau) = 2p(1-p) \sum_{\nu=1}^{K} x_\nu [1-(1-H_K(t)^\nu)] \tag{12}$$

with coefficients a_ν corresponding to (3)

$$a_\nu = \left[2p(1-p) \binom{K}{\nu}^{-1} \right] \sum_{i=\nu}^{K} \binom{i}{\nu} x_i \qquad \nu = 1, \ldots, K. \tag{13}$$

In order that the coefficients a_ν in (3) for the Hamming distance agree for both Kauffman and neural network models up to order K the following matrix equation has to be fulfilled:

$$\left[2p(1-p) \binom{K}{\nu}^{-1} \right] \sum_{i=\nu}^{K} \binom{i}{\nu} x_i = 1 + \sum_{i=1}^{\nu} (-1)^i \binom{i}{\nu} I_i^{(K)} \qquad \nu = 1, \ldots, K. \tag{14}$$

Since (14) contains non-singular upper- and lower-triangular matrices, there exist unique solutions with respect to the quantities x_ν and $I_\nu^{(K)}$, $\nu = 1, \ldots, K$. In particular one finds

$$I_K^{(K)} = 1 - 2p(1-p). \tag{15}$$

Furthermore, one can show that the special hierarchic structure of the integrals $I_\nu^{(K)}$ in (5) admits only positive solutions x_ν satisfying the sum rule $\Sigma_j x_j = 1$. Hence, for any distribution $\rho(c_{ij})$ defining the coefficients a_ν, the quantities x_ν can be determined from (14), whereas for given $x_\nu \in [0, 1]$ the resulting $I_\nu^{(K)}$ may define a parametrised distribution $\rho(c_{ij})$. In this sense the Kauffman model with suitable x_ν agrees with the neural network model defined by a given distribution $\rho(c_{ij})$ or vice versa.

Note that for zero threshold $I_K^{(K)}$ is strictly zero and (15) cannot be satisfied. Consequently, for zero threshold there exists no Kauffman network with the same dynamical evolution of the Hamming distance as the neural network model. On the other hand, the conditions $x_1 = x_2 = \ldots = x_\nu = 0$ for $\nu \in \{1, \ldots, K-1\}$ define generalised Kauffman networks for which no neural network model exists. Thus, within the above definition of the two models there exists a wide parameter space where the Hamming distances of both models are driven by the same dynamics, though there are parameter choices where one cannot find a corresponding neural network or Kauffman model, respectively. Note however, that there might be other physical quantities which behave quite differently within these two models, since the same behaviour with respect to the dynamical evolution of the Hamming distance does not necessarily imply that both systems underlie the same structure.

In summary, we have shown that the above neural network model shares striking common features with the Kauffman model on the level of system behaviour as well as on a formal level. Moreover, on this level we have also demonstrated that there is a close relationship with the Landau description of time-dependent critical phenomena.

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